

# Extended conformal symmetries

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We discuss various aspects of two-dimensional extended conformal symmetries, better known under the name “ $W$ -symmetries”. In particular, we discuss the gauging of  $W$ -symmetries and the construction of the so-called “ $W$ -gravity” theories.

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## 1. Introduction

The role of conformal symmetries in physical theories has a rather long history. In general, the conformal symmetries of a given space are defined as the symmetries which leave the metric of that space invariant up to a scale factor. In a four-dimensional Minkowski spacetime there are fifteen of such symmetries: four translations, six Lorentz rotations, one scale transformation, or dilatation, and four so-called special conformal transformations.

One of the earlier applications of conformal symmetries has been the idea that physics at very high energy should exhibit a conformal invariance, leading to Ward identities. At lower energies some of these conformal symmetries are broken and this leads to a violation of the Ward identities. In particular, the role of scale transformations has been investigated in this context in the beginning of the 70's. For a review of these developments, see ref. [1].

Another application of conformal symmetries has occurred in the study of supergravity. There, the conformal symmetries provide a useful mechanism for constructing different kinds of off-shell formulations. The idea is that a given Poincaré supergravity multiplet decomposes into different conformal multiplets. These conformal multiplets then serve as basic building blocks for the construction of different types of Poincaré supergravity multiplets. For a review, see ref. [2].

Outside four dimensions, the most important applications of conformal symmetries have been in a two-dimensional context. It was Polyakov who made the observation that two-dimensional statistical-mechanical models at their criti-

cal point should be conformally invariant [3]. The important point is that in two dimensions the conformal symmetries do not only include the usual scale transformations but an infinity of other symmetries as well. Due to this fact, two-dimensional conformal symmetries give very stringent restrictions on the form of the correlation functions of these statistical-mechanical models. The infinite number of two-dimensional conformal symmetries are better known under the name Virasoro symmetries and the corresponding infinite-dimensional algebra is called the Virasoro algebra.

Besides statistical-mechanical models, the Virasoro symmetries also play a crucial role in the construction of string theories. String theory can be described by a two-dimensional action involving the coordinates of the string and the two-dimensional metric. Since the string coordinates are scalars from the two-dimensional point of view, one can consider the string action as nothing else than a two-dimensional field theory for a bunch of scalars coupled to two-dimensional gravity. It is well known that such an action is conformally invariant and that is how the Virasoro symmetries enter into the game.

Two-dimensional conformally invariant theories are denoted as Conformal Field Theories (CFT). A lot of progress has been made in recent years in understanding the structure of CFT. An important role in these developments has been played by the Virasoro algebra and its representation theory. The Virasoro generators can be assigned to have conformal spin two. It is then natural to ask oneself the question whether one can extend the Virasoro algebra by the addition of further generators of higher spin. Such extended algebras, if they exist, are usually called “ $W$ -algebras”. Similarly, the corresponding extended conformal symmetries are called “ $W$ -symmetries”.

The study of  $W$ -symmetries can be motivated from several points of view. First of all,  $W$ -symmetries can be used to gain a better understanding of the structure of CFT. For instance, it is known that representations of the Virasoro algebra for values of the central charge  $c \geq 1$  require an infinite number of primary fields [5]. The idea is now that by extending the Virasoro symmetry to a  $W$ -symmetry one can restrict oneself to a *finite* number of “ $W$ -primary” fields. Also, certain exceptional, so-called off-diagonal, modular invariants in CFT can be understood as usual diagonal invariants with respect to a  $W$ -algebra [6].

The second, most important, motivation to investigate  $W$ -symmetries is that they are “natural” symmetries. Roughly speaking, the  $W$ -symmetries form extensions of the Virasoro symmetries in the same way as in group theory  $SU(N)$  transformations form a natural extension of the  $SU(2)$  transformations. When studying  $SU(2)$  group theory, it is natural at some point to extend one’s horizon to  $SU(N)$ . Similarly, in the study of the Virasoro symmetries, one should face sooner or later the extension to  $W$ -symmetries. The fact that  $W$ -symmetries form a natural extension of the Virasoro symmetries can be best seen from the fact that in the last few years a variety of physical models have emerged which

exhibit a  $W$ -symmetry. These models range from Toda field theories and matrix models of two-dimensional gravity to the theory of nonlinear differential equations, like the KP hierarchy.

The third motivation for studying  $W$ -symmetries is that they might be used for the construction of new string models in the same way as the Virasoro symmetries form the starting point for the construction of “ordinary” string models. A first step in this direction is the gauging of  $W$ -symmetries and the construction of the corresponding so-called “ $W$ -gravity” theories. These  $W$ -gravity theories can then be used for the construction of new “ $W$ -string” models.

It is the aim of these lectures to provide a pedagogical introduction into the subject of  $W$ -symmetries. For clarity, and to set up our notation, we will first in section 2 give a brief review of some basic properties of the Virasoro symmetries. In section 3 we will discuss the most simple example of a  $W$ -algebra. This is the so-called  $W_3$ -algebra of ref. [4]. Other  $W$ -algebras will be discussed in section 4. Section 5 deals with the application of  $W$ -symmetries to  $W$ -gravity: we will discuss the gauging of  $W$ -symmetries and the structure of  $W$ -gravity theories. The application of  $W$ -gravity to the construction of  $W$ -string theories can be found in other review articles (see below) and will not be treated here. Finally, in section 6 I will discuss some of the recent developments in the field.

I have tried to keep the overlap with other review articles that have appeared recently in the literature to a minimum. The reader who wants to know more about the subject is invited to consult these articles. For a general introduction into the subject of  $W$ -symmetries, see ref. [7]. A representative list of review articles on  $W$ -gravity and  $W$ -strings can be found in refs. [8–13]. Finally, I should mention that most of the work reported here was done in collaboration with other people. These include: Adel Bilal, Harm-Jan Boonstra, Mees de Roo, Bernard de Wit, Paul Howe, Chris Pope, Larry Romans, Ergin Sezgin, Shawn Shen, Kelly Stelle and Misha Vasiliev. I would like to thank them for the many stimulating discussions I shared with them.

## 2. The Virasoro algebra

The purpose of this section is to give a brief review of some basic properties of the Virasoro symmetries and to familiarize the reader with the notation and conventions which are used throughout these lectures. Due to lack of space I have to be rather schematic. For more information, see, e.g., the review of ref. [14].

We consider a single real free scalar field  $\phi$  corresponding to the action

$$S = \frac{1}{2} \int d^2x \sqrt{-g} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi, \quad (1)$$

where  $x^\mu = (x^0, x^1)$  are the real coordinates of a two-dimensional world sheet.

We will use a Euclidean signature but this is not essential for our purposes. Choosing a conformal gauge  $g_{\mu\nu} = e^{\Omega} \delta_{\mu\nu}$  and using complex coordinates  $z, \bar{z} = x^0 \pm ix^1$ , we can rewrite the action in the following way:

$$S = \frac{1}{2} \int dz d\bar{z} \partial\phi \bar{\partial}\phi. \quad (2)$$

The field equation corresponding to this action reads  $\bar{\partial}\partial\phi = 0$  and the general solution is given by  $\phi(z, \bar{z}) = \phi(z) + \phi(\bar{z})$ , i.e., the field decomposes into a holomorphic part  $\phi(z)$  and an anti-holomorphic part  $\phi(\bar{z})$ <sup>#1</sup>. For simplicity, we will often only consider the holomorphic sector of the theory in our calculations. It is then understood that the same calculation goes through for the anti-holomorphic part.

In a canonical quantisation of the holomorphic part of the theory it is convenient to consider the coordinate  $\bar{z}$  as the evolution parameter and  $z$  as the “space” parameter. The canonical Dirac bracket is given by

$$\{\phi(z), \partial_w \phi(w)\}_{\bar{z}=w} = \delta(z - w). \quad (3)$$

The two-dimensional conformal or Virasoro transformations are given by  $z \rightarrow \epsilon(z)$  and a similar transformation for the anti-holomorphic coordinates. Indeed, one can verify that the action (2) is invariant under the transformation  $\delta\phi = \epsilon(z)\partial\phi$ . This transformation is generated by the holomorphic part  $T(z)$  of the energy-momentum tensor, which is given by

$$T(z) = \frac{1}{2} \partial\phi(z)\partial\phi(z) \quad (4)$$

and is conserved with respect to the anti-holomorphic coordinate  $\bar{z}$ , i.e.,  $\bar{\partial}T = 0$ . Using the basic Dirac bracket (3), one can show that the  $T(z)$  satisfy the following brackets:

$$\{T(z), T(w)\} = \partial_z \delta(z - w) (T(z) + T(w)). \quad (5)$$

In terms of Fourier components one obtains the algebra

$$[L_m, L_n] = (m - n)L_{m+n}. \quad (6)$$

We will call this the “classical” Virasoro algebra to indicate that the algebra can be realised as a bracket algebra<sup>#2</sup>.

In CFT theory one often represents a (Dirac or Poisson) bracket by the singular terms in the operator product expansion (OPE) of the “coordinates”  $\phi$  and

<sup>#1</sup> One might wonder about the reality condition for  $\phi$ . The idea is that effectively we have complexified everything, thereby considering the coordinates  $z$  and  $\bar{z}$  as *independent* variables. One should then also consider the fields  $\phi(z)$  and  $\phi(\bar{z})$  as independent. At the end one imposes the reality condition  $z = \bar{z}^*$  and one is left with one real field  $\phi$ . For more on this, see, e.g., ref. [14].

<sup>#2</sup> Since the brackets refer to a specific free field realisation we should strictly speaking say that the Virasoro algebra is classical with respect to this particular realisation.

the “momenta”  $\partial\phi$ . Using this notation, the Dirac bracket (3) is represented by

$$\phi(z)\partial\phi(w) = -\frac{1}{z-w} + \text{regular part}. \tag{7}$$

Since the scalar  $\phi$  often only occurs via its derivative  $\partial\phi$  we give the basic OPE expansion in terms of  $A \equiv \partial\phi$  and write

$$A(z)A(w) = \frac{1}{(z-w)^2} + \text{regular part}. \tag{8}$$

A bracket thus corresponds to taking a *single* contraction between the  $A$ ’s. Using this notation we can write  $T = \frac{1}{2}AA$  and the classical Virasoro algebra takes the form

$$T(z)T(w) = \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{regular part}. \tag{9}$$

In order to proceed from the classical to the quantum Virasoro algebra one should not only take *single* but also *multiple* contractions between the  $A$ ’s. Furthermore, one should define the normal ordered product of two operators. We use the following natural normal ordering, indicated by round brackets (see, e.g., ref. [6]):

$$(AA)(z) \equiv \lim_{z \rightarrow w} \{A(z)A(w) - \text{singular part}\}. \tag{10}$$

In the case of the one-scalar realisation, we see that by taking multiple contractions between the currents, the classical Virasoro algebra is deformed into the following quantum Virasoro algebra:

$$T(z)T(w) = \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{regular part}, \tag{11}$$

with  $c = 1$ . The current-independent term at the right-hand side represents a so-called central extension of the Virasoro algebra. In Fourier modes the algebra is given by

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}c(m^3 - m)\delta_{m+n,0}. \tag{12}$$

We have seen that a free real scalar leads to a central charge  $c = 1$ . Other values of the central charge can be obtained by adding a so-called background charge  $a$  to the definition of the energy-momentum tensor:

$$T = \frac{1}{2}AA + \sqrt{2}aA. \tag{13}$$

One thus obtains a quantum Virasoro algebra with central charge given by

$$c = 1 - 24a^2. \tag{14}$$

Note that for any value of  $c$  the Virasoro algebra has a  $sl(2)$  subalgebra generated by the Fourier components  $L_{-1}, L_0$  and  $L_1$ . This subalgebra is characterised by the conformal transformations  $\oint dz \epsilon(z)T(z)$  with  $\partial^3\epsilon = 0$ .

### 3. The $W_3$ -algebra

The most simple example of a  $W$ -symmetry is provided by the  $W_3$ -algebra [4]. In this case the Virasoro algebra is extended by a single generator  $W$  of conformal spin three. The classical version of the algebra, indicated by  $w_3$ , is given by

$$\begin{aligned} T(z)T(w) &= \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{regular part}, \\ T(z)W(w) &= \frac{3W(w)}{(z-w)^2} + \frac{\partial W(w)}{z-w} + \text{regular part}, \\ W(z)W(w) &= \frac{2A(w)}{(z-w)^2} + \frac{\partial A(w)}{z-w} + \text{regular part}, \end{aligned} \quad (15)$$

with  $A = TT$ . The  $w_3$ -algebra can be realised as a Poisson bracket algebra in terms of an arbitrary number  $n$  of free scalar fields  $A^i \equiv \partial\phi^i$  ( $i = 0, \dots, n-1$ ) which satisfy the basic OPE's

$$A^i(z)A^j(w) = \frac{g^{ij}}{(z-w)^2} + \text{regular part}, \quad (16)$$

where  $g^{ij}$  are yet undetermined coefficients. A convenient parametrisation of the spin-two and spin-three generators in terms of these scalar fields is given by

$$T = \frac{1}{2}g_{ij}A^iA^j, \quad W = \frac{1}{3}d_{ijk}A^iA^jA^k, \quad (17)$$

where  $d_{ijk}$  are undetermined coefficients and  $g_{ij}$  is the inverse of  $g^{ij}$ . The classical  $w_3$ -algebra is then satisfied if the following identity holds [15]:

$$d_{(ij}{}^m d_{kl)m} = s^2 g_{(ij} g_{kl)}, \quad (18)$$

where  $s$  is an arbitrary parameter which is fixed by the choice of normalisation of the  $W$  generator. It was noted in ref. [16] that this identity also occurs in five-dimensional matter coupled to supergravity theories [17]. For a more recent discussion in this context, see ref. [18]. Besides four "special" solutions to this identity corresponding to  $n = 5, 8, 14$  and  $26$ , there exists a solution for generic values of  $n$  given by

$$d_{000} = s, \quad d_{0\mu\nu} = -s g_{\mu\nu}, \quad (19)$$

where the index  $i$  is split into "0" and an  $(n-1)$ -component index  $\mu$ .

The quantum deformation  $W_3$  of the classical  $w_3$ -algebra (15) takes a more complicated form. Its expression is given by [4]

$$\begin{aligned} T(z)T(w) &= \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{regular part}, \\ T(z)W(w) &= \frac{3W(w)}{(z-w)^2} + \frac{\partial W(w)}{z-w} + \text{regular part}, \end{aligned}$$

$$\begin{aligned}
 W(z)W(w) = & \frac{c}{3(z-w)^6} + \frac{2T(w)}{(z-w)^4} + \frac{\partial T(w)}{(z-w)^3} \\
 & + \frac{3}{10} \frac{\partial^2 T(w)}{(z-w)^2} + \frac{1}{15} \frac{\partial^3 T(w)}{z-w} \\
 & + \frac{16}{22+5c} \left( \frac{2A(w)}{(z-w)^2} + \frac{\partial A(w)}{z-w} \right) + \text{regular part},
 \end{aligned}
 \tag{20}$$

with  $A = (TT) - \frac{3}{10} \partial^2 T$ .

In order to find realisations of the quantum  $W_3$ -algebra, the  $T$  and  $W$  generators are parametrised as follows [15,16]:

$$T = \frac{1}{2} g_{ij} (A^i A^j) + \sqrt{2} a_i A^{i'}, \tag{21}$$

$$W = \frac{1}{3} d_{ijk} (A^i A^j A^k) + 2\sqrt{2} e_{ij} (A^i A^{j'}) + 2f_i A^{i'}, \tag{22}$$

where  $g_{ij}, a_i, d_{ijk}, e_{ij}$  and  $f_i$  are yet undetermined coefficients. Note that with the above ansatz the spin-two generator  $T(z)$  already satisfies the Virasoro algebra with central charge  $c = n - 24a_i a^i$ . The requirement that this ansatz for the generators satisfies the algebra (20) leads to a system of five equations for the unknown coefficients. From it, one can derive the following  $n$ -scalar realisation [16]:

$$T = \frac{1}{2} (A_0 A_0) + \sqrt{2} a_0 A'_0 + T_\mu, \tag{23}$$

$$W = -\frac{1}{3} (A_0 A_0 A_0) - \sqrt{2} a_0 (A_0 A'_0) - \frac{2}{3} a_0^2 A''_0 + 2(A_0 T_\mu) + \sqrt{2} a_0 T'_\mu,$$

Note that only the free scalar  $\phi^0$  occurs explicitly in the above realisation. The other  $n - 1$  scalars are represented by  $T_\mu$ , which commutes with  $A_0$  and satisfies a Virasoro algebra with central charge given by  $c_\mu = \frac{1}{4}c + \frac{1}{2}$ . The background charge  $a_0$  is related to the central charge parameter  $c$  via  $c = 2(1 - 16a_0^2)$ . The resulting realisation coincides for  $n = 2$  with the Fateev-Zamolodchikov (FZ) two-scalar realisation [19]. It can be viewed as a natural generalisation of the FZ realisation to an arbitrary number  $n$  of scalar fields.

#### 4. Other $W$ -algebras

The  $W_3$ -algebra is the most simple example of a higher-spin extension of the Virasoro algebra. More complicated examples exist as well. One can divide them into two classes. The first class consists of algebras which only contain a *finite* number of higher-spin generators of maximum spin  $s = N$ . Such algebras are generically denoted as  $W_N$ -algebras. They were first discussed in refs. [19,20]. They all share the property with the  $W_3$ -algebra that they are *nonlinear* algebras: The OPE of two generators is in general a polynomial in the generators. The  $W_N$ -algebras are therefore no Lie algebras of the ordinary type. If the algebra

contains one or more supersymmetry generators of conformal spin  $s = 3/2$  this is often specified by calling the algebra a super- $W_N$ -algebra.

The other class of  $W$ -algebras to consider are the ones which contain an *infinite* number of higher-spin generators with, often, each spin occurring once. These algebras are generically denoted as  $W_\infty$ -algebras and were first discussed in ref. [21]. The supersymmetric ones are called super- $W_\infty$ -algebras. In contrast to the  $W_N$ -type algebras the  $W_\infty$ -algebras are linear, i.e., they are (infinite-dimensional) Lie algebras. The reason of this difference is easy to understand. Since the  $W_N$ -algebras contain a finite number of generators, any expression in an OPE that has spin higher than the maximum spin  $s = N$  must be expressed as a polynomial in the finite number of generators since generators with spin higher than  $N$  do not occur, by assumption. This is the origin of the nonlinear structure of the  $W_N$ -algebra. In case of the  $W_\infty$ -algebras such nonlinearities can be avoided due to the infinite number of generators with ever increasing spin.

In this section we will treat an example of a  $W_\infty$ -type algebra. The example is the  $N = 2$  super- $W_\infty$ -algebra [22], where the  $N = 2$  indicates that there are two supersymmetries. It is a supersymmetric extension of the  $W_\infty$ -algebra of ref. [21]. We will discuss this particular algebra in much analogy with the  $W_3$ -algebra of the previous section.

The classical  $N = 2$  super- $w_\infty$ -algebra [23] is generated by an infinite set of currents  $w^{(s)}$  of (super-)conformal spin  $s = 1/2, 1, 3/2, 2, \dots$ . Each generator is a superfield depending on the superspace coordinates  $Z = (z, \theta)$ . Each superfield contains two conformal field components of spin  $s$  and  $s + 1/2$ . The  $N = 2$  super- $w_\infty$ -algebra is an extension of the  $N = 2$  super-Virasoro algebra which is generated by  $\{w^{(1)}, w^{(3/2)}\}$ :

$$\begin{aligned}
 w^{(1)}(1)w^{(1)}(2) &= -2\frac{\theta_{12}w^{(3/2)}}{z_{12}} + \text{regular part}, \\
 w^{(3/2)}(1)w^{(1)}(2) &= \frac{\theta_{12}w^{(1)}}{z_{12}^2} - \frac{1}{2}\frac{D_2w^{(1)}}{z_{12}} \\
 &\quad + \frac{\theta_{12}\partial_2w^{(1)}}{z_{12}} + \text{regular part}, \quad (24) \\
 w^{(3/2)}(1)w^{(3/2)}(2) &= \frac{3}{2}\frac{\theta_{12}w^{(3/2)}}{z_{12}^2} - \frac{1}{2}\frac{D_2w^{(3/2)}}{z_{12}} \\
 &\quad + \frac{\theta_{12}\partial_2w^{(3/2)}}{z_{12}} + \text{regular part},
 \end{aligned}$$

where  $\partial = \partial_z, \theta_{12} = \theta_1 - \theta_2, z_{12} = z_1 - z_2 + \theta_{12}$  and the superspace differential operator  $D$  is defined by

$$D = \partial/\partial\theta - \theta\partial. \quad (25)$$

Furthermore, we have used the short-hand notation  $w^{(s)}(1)$  to indicate



$w^{(s)}(Z_1)$ . The full algebra is given by

$$w^{(s)}(1)w^{(t)}(2) = -2 \frac{\theta_{12} w^{(s+t-1/2)}}{z_{12}} + \text{regular part} \tag{26}$$

for  $s$  and  $t$  integer and by

$$w^{(s)}(1)w^{(t)}(2) = (-)^{|2s+1|_2} \left\{ (s+t-\frac{3}{2}) \frac{\theta_{12} w^{(s+t-3/2)}}{-z_{12}^2} - \frac{1}{2} \frac{D_2 w^{(s+t-3/2)}}{z_{12}} + (s-\frac{1}{2}) \frac{\theta_{12} \partial_2 w^{(s+t-3/2)}}{z_{12}} \right\} + \text{regular part} \tag{27}$$

in all other cases with  $s, t \neq 1/2$ . The OPE's where  $s$  and/or  $t$  equals  $1/2$  can be found in ref. [25]. By definition, the symbol  $|s|_2$  is equal to zero for  $s$  even and 1 for  $s$  odd.

The classical  $N = 2$  super- $w_\infty$ -algebra can be realised as a Poisson bracket algebra in terms of two real scalar superfields  $\phi$  and  $\bar{\phi}$  whose basic OPE are given by

$$\phi(1)\bar{\phi}(2) = -\ln z_{12} + \text{regular part}. \tag{28}$$

The fact that two scalar superfields are needed is due to the  $N = 2$  supersymmetry, whose realisation requires at least two  $N = 1$  superfields. In terms of  $\phi$  and  $\bar{\phi}$  the generators are given by

$$w^{(s)} = (\partial\phi)^{s-1} D\phi D\bar{\phi} \quad (s \text{ integer}), \tag{29}$$

$$w^{(s)} = (\partial\phi)^{s-1/2} D\bar{\phi} + \frac{1}{2} D\{D\phi(\partial\phi)^{s-3/2} D\bar{\phi}\} \quad (s \text{ half-integer}).$$

The quantum deformation  $N = 2$  super- $W_\infty$  of the classical  $N = 2$  super- $w_\infty$ -algebra takes a rather complicated form. Its generic expression is given by [24,25]

$$W_\lambda^{(s)}(1)W_\lambda^{(t)}(2) = \sum_{u=1/2}^{s+t-1/2} f_{st}^u(D_1, D_2; \lambda) \frac{\theta_{12} W_\lambda^{(s+t-u)}(2)}{z_{12}} + c(s, t; \lambda) \frac{\theta_{12}^{|2(s+t)|_2}}{(z_{12})^{s+t+1/2|2(s+t)|_2}} + \text{regular part}. \tag{30}$$

The structure functions  $f_{st}^u(D_1, D_2; \lambda)$  are polynomials in the supercovariant derivatives of degree  $2u - 1$ , whose explicit form is given in ref. [25]. The same reference also gives an explicit expression for the central charges  $c(s, t; \lambda)$ . The arbitrary parameter  $\lambda$  is related to a choice of basis of the algebra [24]. In terms of  $\lambda$  the central charge  $c(3/2, 3/2; \lambda)$  of the Virasoro subalgebra is given by

$$c = -12(\lambda - 1/4). \tag{31}$$

The main difference in the structure of the classical and quantum algebra is that in the classical case the OPE of two currents of spin  $s$  and  $t$  contains

only one current with maximum spin  $s_{\max}$  whereas in the quantum case the OPE contains lower-spin currents with  $1/2 \leq s \leq s_{\max}$  as well. These lower-spin currents should be considered on the same footing as the central extension of the Virasoro algebra, which can be viewed as a spin-zero generator.

The generators of the quantum algebra can again be given in terms of the scalar superfields  $\phi$  and  $\bar{\phi}$ . The corresponding expressions are certain polynomials in  $\phi$  and  $\bar{\phi}$ . It turns out that it is more convenient to give the expressions in terms of a supersymmetric  $BC$  system. The  $B, C$  superfields are related to the  $\phi, \bar{\phi}$  superfields via the superbosonisation rule [26]

$$B = e^{\phi}, \quad C = e^{-\phi} D\bar{\phi}. \quad (32)$$

In terms of the  $B, C$  superfields the expressions for the quantum generators take the following bilinear form [24]:

$$W_{\lambda}^{(s)} = \sum_{i=0}^{2s-1} \tilde{A}^i(s, \lambda) (D^i B) (D^{2s-i-1} C), \quad (33)$$

where the  $\tilde{A}^i(s, \lambda)$  are certain coefficients whose explicit form is given in refs. [24,25].

## 5. $W$ -gravity

When discussing “ $W$ -gravity”, it is instructive to first consider ordinary two-dimensional gravity and its relation with the Virasoro algebra. Our starting point is the action (1) corresponding to a free scalar field  $\phi$  coupled to gravity. Note that the two-dimensional metric field  $g_{\mu\nu}$  occurs nonpolynomially in the action. However, the corresponding nonlinearities can be understood from a geometric point of view: the action (1) is invariant under arbitrary parametrisations of the two-dimensional world sheet and the corresponding gauge field is  $g_{\mu\nu}$ , which has the geometric interpretation of being the metric tensor of the two-dimensional world sheet.

Since the action (1) is invariant under the conformal scale transformations  $g_{\mu\nu} \rightarrow \lambda g_{\mu\nu}$ , effectively only two of the three components of  $g_{\mu\nu}$  occur in (1). This can be made more explicit by going to the complex basis described after (1) and substituting the following parametrisation of the metric tensor:

$$g_{\mu\nu} = e^{\Omega} \begin{pmatrix} 2\bar{h} & 1 + h\bar{h} \\ 1 + h\bar{h} & 2h \end{pmatrix}. \quad (34)$$

One thus obtains the following expression for the Lagrangian:

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} \frac{1}{1 - h\bar{h}} (\partial\phi - \bar{h}\bar{\partial}\phi) (\bar{\partial}\phi - h\partial\phi) \\ &= \frac{1}{2} \partial\phi\bar{\partial}\phi - \frac{1}{2} h\partial\phi\partial\phi - \frac{1}{2} \bar{h}\bar{\partial}\phi\bar{\partial}\phi + h\bar{h}\bar{\partial}\phi\partial\phi + \dots \end{aligned} \quad (35)$$

From this expression we see that only the  $h$  and  $\bar{h}$  components of the metric occur. Furthermore all nonlinear terms in  $h$  and  $\bar{h}$  contain both  $h$  as well as  $\bar{h}$ . This means that all nonlinearities disappear as soon as we impose the gauge condition  $\bar{h} = 0$  (the chiral gauge) or  $h = 0$  (the anti-chiral gauge).

We first discuss how the result in the chiral gauge can be obtained by “chiral gauging” of the free Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial \phi \bar{\partial} \phi. \tag{36}$$

Clearly, the action corresponding to this Lagrangian is invariant under the Virasoro transformation  $\delta \phi = \epsilon \partial \phi$ , where the parameter  $\epsilon$  only depends on  $z$ , i.e.,  $\bar{\partial} \epsilon = 0$  or  $\epsilon = \epsilon(z)$ . The gauging is achieved by the requirement that the action is also invariant if the parameter depends on both  $z$  as well as  $\bar{z}$ , i.e.,  $\bar{\partial} \epsilon \neq 0$  or  $\epsilon = \epsilon(z, \bar{z})$ . To this end we introduce a gauge field  $h$  which transforms inhomogeneously under  $\epsilon$  as  $\delta h = \bar{\partial} \epsilon + \dots$ . The remarkable thing is that, if one now applies the Noether procedure in order to obtain a gauge-invariant action, the procedure stops after the first step, which corresponds to adding to the free action a term of the form gauge field times current. The final result thus takes the form

$$\mathcal{L} = \frac{1}{2} \partial \phi \bar{\partial} \phi - h T, \tag{37}$$

where  $T$  is the energy–momentum tensor given by (4). The transformation rules of  $\phi$  and  $h$  that leave the action invariant are given by

$$\delta \phi = \epsilon \partial \phi, \quad \delta h = \bar{\partial} \epsilon + \epsilon (\partial h) - h (\partial \epsilon). \tag{38}$$

One can show that the above phenomenon, namely that in order to go from the ungauged action to the gauged action it is enough to simply add gauge field times current terms to the free action, holds for any closed algebra (linear or nonlinear) generated by a set of currents. More precisely, the following theorem holds (see also ref. [8]):

**Theorem 1.** *Suppose we have a set of currents  $\{W^A\}$  which form a closed chiral bracket algebra, i.e.,*

$$\{W^A, W^B\} = f^{AB}{}_C(W) W^C \tag{39}$$

*with  $\bar{\partial} W^A = 0$ . Then the following action:*

$$\mathcal{L} = \frac{1}{2} \partial \phi \bar{\partial} \phi - h_A W^A, \tag{40}$$

*is invariant under the transformations*

$$\delta \phi = \{\epsilon_A W^A, \phi\}, \quad \delta h_A = \bar{\partial} \epsilon_A + \Delta h_A, \tag{41}$$

*with  $\Delta h_A = -h_B \epsilon_C f^{CB}{}_A$ .*

The proof of this theorem is rather straightforward. First of all, by construction, the variation of the kinetic term cancels against the inhomogeneous variation of

the gauge field in the gauge field times current terms. The homogeneous variation of these gauge field times current terms, on the other hand, is given by

$$\begin{aligned} \delta(h_A W^A) &= \Delta h_A W^A + h_A \{\epsilon_B W^B, W^A\} \\ &= \Delta h_A W^A + h_A \epsilon_B f^{BA}{}_C W^C. \end{aligned} \tag{42}$$

Requiring this variation to be zero leads to the solution  $\Delta h_A = -h_B \epsilon_C f^{CB}{}_A$  as stated in the theorem.

The nice thing about this theorem is its general applicability. One cannot only use it to gauge linear algebras such as the Virasoro algebra or the (super-)  $W_\infty$ -algebras but also to gauge nonlinear algebras such as the classical  $w_N$ -algebras. The chiral gauging of the classical  $w_3$ -algebra was first given in ref. [15]. Since these gaugings have already been discussed at other places in the literature, we will not consider them here further. The general idea can be explained on the example of the usual two-dimensional gravity case.

Instead of taking the chiral gauge  $\bar{h} = 0$ , one could also consider the anti-chiral gauge  $h = 0$ . The above analysis would go through similarly, with everywhere  $\partial$  replaced by  $\bar{\partial}$ ,  $h$  by  $\bar{h}$ ,  $\epsilon$  by  $\bar{\epsilon}$ , etc. For instance, the Virasoro transformations would now be given by  $\delta\phi = \bar{\epsilon}\bar{\partial}\phi$ . Before gauging  $\bar{\epsilon}$  only depends on  $\bar{z}$ , after gauging it depends on both  $\bar{z}$  and  $z$ .

We now consider the nonchiral gauging where both  $h$  and  $\bar{h}$  are nonzero. We again treat the case of ordinary two-dimensional gravity in detail to explain the general principle. In the case of ordinary gravity we know already the answer: the action is given by (35). In order to generalise this to the case of  $W$ -algebras it is useful to rewrite (35) in the following way [27]:

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2}\partial\phi\bar{\partial}\phi - J\bar{J} + J\bar{\partial}\phi + \bar{J}\partial\phi - \frac{1}{2}hJ^2 - \frac{1}{2}\bar{h}\bar{J}^2 \\ &= -\frac{1}{2}\partial\phi\bar{\partial}\phi - J\bar{J} + J\bar{\partial}\phi + \bar{J}\partial\phi - hT(\partial\phi \rightarrow J) - \bar{h}\bar{T}(\bar{\partial}\phi \rightarrow \bar{J}), \end{aligned} \tag{43}$$

where  $T(\partial\phi \rightarrow J)$  means: take the free-scalar realisation (4) and replace  $\partial\phi$  by  $J$ , and similarly for the barred quantities. Here  $J$  and  $\bar{J}$  are two auxiliary fields which define the following set of “nested covariant derivatives” [28]:

$$J = \partial\phi - \bar{h}\bar{J}, \quad \bar{J} = \bar{\partial}\phi - hJ. \tag{44}$$

One can iteratively solve these equations as follows:

$$\begin{aligned} J &= \partial\phi - \bar{h}\bar{\partial}\phi + h\bar{h}\partial\phi + \dots, \\ \bar{J} &= \bar{\partial}\phi - h\partial\phi + h\bar{h}\bar{\partial}\phi + \dots. \end{aligned} \tag{45}$$

Substituting these expressions back into the action (43) reproduces the form of the action given in (35). The nice effect of the auxiliary fields is that effectively they give a complete split of the left-movers and the right-movers of the action. It is as if one has to do the chiral gauging twice, for each sector separately.

Like in the chiral case, one can show that the nonchiral gauging of two-dimensional gravity discussed in this section can be generalised to any  $W$ -algebra which has a bracket realisation of the form (39). The nonchirally gauged action is given by the same formula (43) but instead in the last line one has to sum over all gauge field times current terms and in all the currents one has to replace  $\partial\phi$  by  $J$  and  $\bar{\partial}\phi$  by  $\bar{J}$ . This then gives us the definition of  $W$ -gravity as a higher-spin extension of ordinary two-dimensional gravity. The nonchiral gauging of the  $w_3$ -algebra was first given in ref. [28].

We now briefly discuss the extension from classical to quantum  $W$ -gravity. For more details, see, e.g., the review articles [8–13]. When quantising ordinary gravity one expects in general a conformal anomaly. The coefficient of the anomaly is related to the central extension  $c$  of the Virasoro algebra. The conformal anomaly is absent if the total central charge is zero. The total central charge receives contributions from the matter sector as well as the ghosts which are needed to gauge fix the Virasoro symmetries. Since the ghosts contribute  $-26$  the matter sector must contribute  $+26$  and that is why a critical string requires 26 free scalars. In the case of the  $W$ -algebras a similar thing happens. However, also anomalies of a new type arise, which are absent in the Virasoro case.

This comes about as follows. In ordinary gravity the conformal anomaly is related to the central charge, which may be viewed as the occurrence of a spin-zero generator in the algebra. The presence of this lower-spin generator is a manifestation of the anomaly. In the case of the  $W$ -algebras the same thing happens but besides spin-zero generators other lower-spin generators of nonzero spin arise as well. Compare, for instance, the classical  $w_3$ -algebra (15) with the quantum  $W_3$ -algebra (20). In the classical case the spin-three generators only give rise to the composite spin-four generator. In the quantum case, however, an additional spin-two and an additional spin-zero generator (the central extension) show up. The central extension corresponds to the usual conformal anomaly. The extra spin-two generator corresponds to another, so-called matter-dependent, anomaly. These anomalies were first discussed in ref. [29]. In an anomaly-free  $W$ -gravity theory these anomalies can be get rid of by appropriate renormalisations of the currents. The renormalised currents form a realisation of the quantum  $W$ -algebra, which in general is a deformation of the classical  $W$ -algebra. A nice way of summarising the requirement that all anomalies cancel is to say that the BRST operator corresponding to the  $W$ -algebra must be nilpotent [31]. Instead of cancelling the conformal anomaly, one could also consider so-called induced  $W$ -gravity actions. For a discussion of this approach, see ref. [9].

To give an example of how the above-mentioned renormalisations come about consider the  $N = 2$  super- $W_\infty$ -algebra treated in section 4. A similar discussion for the bosonic case can be found in ref. [30]. The first few classical currents,

in the  $\lambda = 0$  basis, are given by

$$\begin{aligned} w^{(1/2)} &= D\bar{\phi}, \\ w^{(1)} &= D\phi D\bar{\phi}, \\ w^{(3/2)} &= \frac{1}{2}\partial\phi D\bar{\phi} + \frac{1}{2}D\phi\partial\bar{\phi}, \\ &\vdots \end{aligned} \tag{46}$$

The requirement of cancelling all anomalies then leads to the following renormalisations (the corresponding Feynman diagram calculation can be found in ref. [25]):

$$\begin{aligned} W^{(1/2)} &= D\bar{\phi}, \\ W^{(1)} &= D\phi D\bar{\phi} + \sqrt{\hbar}\partial\phi, \\ W^{(3/2)} &= \frac{1}{2}\partial\phi D\bar{\phi} + \frac{1}{2}D\phi\partial\bar{\phi} + \frac{1}{2}\sqrt{\hbar}\partial D\phi, \\ &\vdots \end{aligned} \tag{47}$$

They are exactly the bosonised version of the first few currents,

$$\begin{aligned} W^{(1/2)} &= BC, \\ W^{(1)} &= (DB)C, \\ W^{(3/2)} &= \frac{1}{2}(\partial B)C - \frac{1}{2}(DB)(DC), \\ &\vdots \end{aligned} \tag{48}$$

generating the quantum  $N = 2$  super- $W_\infty$ -algebra [cf. (33)].

## 6. Miscellaneous

Having an anomaly-free  $W$ -gravity theory at our disposal, it is natural to ask oneself the question whether it might be used to construct a “ $W$ -string” model in the same way as ordinary two-dimensional gravity is used as a starting point for ordinary string theories. This possibility was already mentioned sometime ago in ref. [32], where it was also suggested that the spectrum of such a  $W$ -string should contain higher-spin massless states. More recently, this question was reconsidered in refs. [33,34]. In particular, the outcome of the analysis of ref. [34] seems to yield the result that in the massless sector the spectrum of the  $W$ -string is the same as that of the ordinary string. It would be interesting to see whether one could give a somewhat firmer basis to the concept of a  $W$ -string by working out in more detail some of its properties, like its interactions, a “ $W$ -geometric” interpretation, the mathematical properties of the moduli space

of  $W$ -gravity etc. <sup>#3</sup> Very recently, interesting new developments have also occurred in the discussion of noncritical  $W$ -strings [37]. We will not discuss these issues further here but we expect (and hope) that more results on  $W$ -strings will be obtained in the not too distant future. Instead we would like to briefly discuss two new results in which we have been involved ourselves.

The first result is related to the question of finding new realisations of  $W$ -symmetries. Recently, we investigated in ref. [36] this question for the simplest case of the  $W_3$ -algebra, thereby generalising an earlier analysis of ref. [16]. The generalisation is obtained by allowing a spin-four operator with norm zero in the OPE of two spin-three currents. To be more precise, instead of the third equation of (20) we require that the following OPE holds:

$$W(z)W(w) = \text{as in (20)} + \frac{V(w)}{(z-w)^2} + \frac{\frac{1}{2}\partial V(w)}{z-w}, \tag{49}$$

where  $V$  is a spin-four null operator, i.e.,  $\langle VV \rangle = 0$ . Of course, strictly speaking, the algebra corresponding to (49) is not the same as the  $W_3$ -algebra given in (20). However, since  $V$  is a null operator, it can only generate other null fields in its OPE. The full set of null operators constitutes an ideal of the algebra. It is therefore consistent to set all these null operators equal to zero and one thus obtains a representation of the  $W_3$ -algebra.

The analysis of ref. [16] now changes in the sense that instead of (18) the following equation should hold:

$$\begin{aligned} &24S^{ijkl}S_{ijkl} + 30S_i^{ikl}S_j^j{}_{kl} - 280S_i^{ikl}S_j^j{}_k a_l a_m \\ &- 60\sqrt{2}S_i^{ikl}T_{klm}a^m + 24\sqrt{2}S_i^{ikl}T_{mkl}a^m \\ &+ \frac{560}{3}\sqrt{2}S_i^{ikl}(T_{mnl} + 2T_{lmn})a_k a^m a^n - 12T^{ijk}T_{ijk} \\ &- 16T^{ijk}T_{ikj} + 60T_{ijk}T^{ij}{}_l a^k a^l - 48T_{ijk}T_l{}^{ij} a^k a^l \\ &+ \frac{328}{3}T_{kij}T_l{}^{ij} a^k a^l + \frac{104}{3}T_{kij}T_l{}^{ji} a^k a^l \\ &- \frac{560}{9}(T_{ijm}T_{kl}{}^m + 4T_{imj}T_{kl}{}^m + 4T_{imj}T_{k'l}{}^m)a^i a^j a^k a^l = 0, \end{aligned} \tag{50}$$

where  $S$  and  $T$  are given by

$$S_{ijkl} = d_{(ij}{}^m d_{kl)m} - \frac{24N_3}{c(22 + 5c)}g_{(ij}g_{kl)}, \tag{51}$$

$$T_{ijk} = 4\sqrt{2}\left(-2d_{ij}{}^l e_{[kl]} + 2e_{(i}{}^l d_{j)kl} - \frac{24N_3}{c(22 + 5c)}g_{ij}a_k\right), \tag{52}$$

and the coefficient  $N_3$  is the norm of the spin-three generator, i.e.,  $\langle WW \rangle = N_3$  (for more details, see ref. [36]).

<sup>#3</sup> See ref. [35] for a recent discussion of the latter point.

This allows us to take the following less restrictive ansatz for the coefficients  $d_{ijk}$ :

$$d_{000} = s, \quad d_{0\mu\nu} = t g_{\mu\nu}, \quad (53)$$

with  $s$  and  $t$  free parameters (although one of them may be fixed by choosing a normalisation for  $W$ ).

For two scalars we were able to give a complete classification of all possible realisations. Besides the solution of ref. [16], which for two scalars reduces to the solution of ref. [19], we found four more solutions. Firstly, there are two inequivalent solutions at central charge  $c = -2$ . They are given by

$$\begin{aligned} T &= \frac{1}{2}(A_0 A_0) + \frac{1}{2}(A_1 A_1) + \frac{1}{2}A'_0 + \frac{1}{6}\sqrt{3}A'_1, \\ W &= \frac{5}{9}(A_0 A_0 A_0) + \frac{5}{6}v(A_0 A'_0) + \frac{5}{36}A''_0 \\ &\quad + (A_0 A_1 A_1) + \frac{1}{3}\sqrt{3}(A_0 A'_1) + \frac{1}{2}(A_1 A'_1) + \frac{1}{12}\sqrt{3}A''_1, \end{aligned} \quad (54)$$

and

$$\begin{aligned} T &= \frac{1}{2}(A_0 A_0) + \frac{1}{2}(A_1 A_1) + \frac{1}{3}\sqrt{3}A'_1, \\ W &= \frac{5}{9}(A_0 A_0 A_0) + (A_0 A_1 A_1) + \frac{1}{2}\sqrt{3}(A_0 A'_1) \\ &\quad + \frac{1}{6}\sqrt{3}(A'_0 A_1) + \frac{1}{18}A''_0, \end{aligned} \quad (55)$$

respectively. Secondly, there are two other solutions at  $c = 4/5$ . They are given by

$$\begin{aligned} T &= -\frac{1}{2}(A_0 A_0) + \frac{1}{2}(A_1 A_1) + \frac{1}{2}\sqrt{6}A'_0 + \frac{2}{3}\sqrt{10}A'_1, \\ W &= \frac{13}{15}(A_0 A_0 A_0) - \frac{13}{10}\sqrt{6}(A_0 A'_0) + \frac{13}{10}A''_0 \\ &\quad - (A_0 A_1 A_1) - \frac{4}{5}\sqrt{10}(A_0 A'_1) + \frac{1}{2}\sqrt{6}(A_1 A'_1) + \frac{1}{5}\sqrt{60}A''_1, \end{aligned} \quad (56)$$

and

$$\begin{aligned} T &= \frac{1}{2}(A_0 A_0) + \frac{1}{2}(A_1 A_1) + \frac{1}{10}\sqrt{10}A'_1, \\ W &= \frac{13}{15}(A_0 A_0 A_0) + (A_0 A_1 A_1) + \frac{1}{2}\sqrt{10}(A_0 A'_1) \\ &\quad - \frac{3}{10}\sqrt{10}(A'_0 A_1) - \frac{1}{10}A''_0. \end{aligned} \quad (57)$$

The second and fourth realisations also occur in ref. [38] as specific truncations of a nonlinear  $W_\infty$ -algebra. It remains to be investigated whether the first and third realisations follow from other construction procedures as well.

The second result we would like to briefly discuss here has to do with an interesting relation between  $W_\infty$ -symmetries and self-dual gravity in  $2 + 2$  dimensions [39]. Self-dual gravity theories in  $2 + 2$  dimensions also occur as the low-energy limit of the  $N = 2$  superstring [40]. Recently, a supersymmetric version of a self-dual gravity theory in  $2 + 2$  dimensions has been constructed [41]. Very recently, we constructed a new self-dual supergravity theory [42].



Our starting point is the new minimal formulation of ref. [43]. The basic observation is that the supergravity transformation rules can be written in the following Yang–Mills-like form [44,45]:

$$\begin{aligned}\delta\Omega_{\mu-}{}^{ab} &= \frac{1}{2}\bar{\epsilon}_+\gamma_\mu\psi_-^{ab} + \frac{1}{2}\bar{\epsilon}_-\gamma_\mu\psi_+^{ab}, \\ \delta\psi_\pm^{ab} &= -\frac{1}{4}\gamma^{cd}\hat{R}_{cd}{}^{ab}(\Omega_-)\epsilon_\pm \mp \hat{F}^{ab}(V_+)\epsilon_\pm, \\ \delta\hat{F}^{ab}(V_+) &= \frac{1}{4}\bar{\epsilon}_+\gamma^\mu D_\mu(\Omega_-)\psi_-^{ab} + \frac{1}{4}\bar{\epsilon}_-\gamma^\mu D_\mu(\Omega_-)\psi_+^{ab},\end{aligned}\tag{58}$$

where  $D_\mu(\Omega_-)$  is the supercovariant derivative and the  $\Omega_-$  covariantization in the last line acts on the fermionic as well as the vectorial indices of  $\psi_\pm^{ab}$ . Furthermore,

$$\begin{aligned}\Omega_{\mu\pm}{}^{ab} &= \omega_\mu{}^{ab}(e, \psi) \pm \hat{H}_\mu{}^{ab}, \\ V_{\mu+} &= V_\mu + \frac{1}{6}\epsilon_\mu{}^{abc}\hat{H}_{abc}.\end{aligned}\tag{59}$$

We see that the role of the Yang–Mills group is played by the Lorentz group, which in 2 + 2 dimensions is  $\text{SO}(2, 2) = \text{SO}(2, 1) \otimes \text{SO}(2, 1)$ . This suggests to impose the following self-duality condition, which effectively eliminates one of the  $\text{SO}(2, 1)$  factors of the  $\text{SO}(2, 2)$  Lorentz group:

$$\begin{aligned}\psi_\pm^{ab} &= \frac{1}{2}\epsilon^{abcd}\psi_{cd\pm}, \\ \hat{R}_{cd}{}^{ab}(\Omega_-) &= \frac{1}{2}\epsilon^{abef}\hat{R}_{def}(\Omega_-), \\ \hat{F}^{ab}(V_+) &= \frac{1}{2}\epsilon^{abcd}\hat{F}_{cd}(V_+).\end{aligned}\tag{60}$$

These self-duality equations are special solutions to the field equations corresponding to the following  $R^2$ -type action:

$$\begin{aligned}e^{-1}\mathcal{L}(R^2) &= \frac{1}{4}R_{\mu\nu}{}^{ab}(\Omega_-)R^{\mu\nu ab}(\Omega_-) \\ &\quad - 2\hat{F}_{ab}(V_+)\hat{F}^{ab}(V_+) + \frac{1}{2}\bar{\psi}^{ab}\gamma^\mu D_\mu\psi_{ab} \\ &\quad + \frac{1}{8}\bar{\psi}_\mu\gamma^{cd}\gamma^\mu\psi_{ab}(R_{cd}{}^{ab}(\Omega_-) + \hat{R}_{cd}{}^{ab}(\Omega_-)),\end{aligned}\tag{61}$$

where  $\psi_\mu = \psi_{\mu+} + \psi_{\mu-}$ . For more details, see ref. [42].

The interesting new feature of this new self-dual supergravity theory is that we are now dealing with a *torsionful* Riemann curvature. The self-duality equation (60) then does *not* imply that the Ricci tensor vanishes. Instead, the Ricci tensor is proportional to torsion-dependent terms. It is interesting to investigate the implications of this fact.

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